Online Appendix to
"The Impact of Competition on Prices with Numerous Firms"

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This online appendix discusses some additional issues. Appendix B contains details and derivations for the Perloff-Salop, Sattinger, and Hart models. Appendix C endogenizes the degree of product differentiation / obfuscation. Appendix D contains proofs that are omitted from the main paper. Appendix E derives second-order equilibrium conditions for the Perloff-Salop, Sattinger and Hart models.

Appendix B   Random-Utility Models

This section provides details for the derivation of the markup expressions for the three random utility models analyzed in this paper.

Perloff-Salop

Recall from (4) that in the Perloff-Salop model, the demand function for good i is the probability that difference between the demand shock and the price is maximized by good i:

$$D\left(p_1, ..., p_n; i\right) = \mathbb{P}\left(X_i - p_i \geq \max_{j \neq i} X_j - p_j\right)$$

$$= \mathbb{E}_{X_i}\left[\prod_{j \neq i} \mathbb{P}\left(x - p_i \geq X_j - p_j \mid X_i = x\right)\right]$$

$$= \mathbb{E}_{X_i}\left[\prod_{j \neq i} F\left(x - p_i + p_j\right)\right]$$

$$= \int_{w_l}^{w_u} f\left(x\right) \prod_{j \neq i} F\left(x - p_i + p_j\right) dx.$$

Using $D(p_i, p; n)$ to denote the demand for good i at price p_i when all other firms set price p and using $D_1(p_i, p; n)$ to denote $\partial D(p_i, p; n)/\partial p_i$, we may calculate

$$D(p_i, p; n) = \int_{w_l}^{w_u} f(x) F^{n-1}(x - p_i + p) \, dx$$

$$D_1(p_i, p; n) = -(n-1) \int_{w_l}^{w_u} f(x) f(x - p_i + p) F^{n-2}(x - p_i + p) \, dx.$$

Note that in a symmetric equilibrium

$$D(p, p; n) = \int_{w_l}^{w_u} f(x) F^{n-1}(x) \, dx = 1/n,$$

$$D_1(p, p; n) = -(n-1) \int_{w_l}^{w_u} f^2(x) F^{n-2}(x) \, dx.$$

It follows that the Perloff-Salop markup μ_n^{PS} is

$$p - c = -\frac{D(p, p; n)}{D_1(p, p; n)} = \frac{1}{n(n-1) \int_{w_l}^{w_u} f^2(x) F^{n-2}(x) \, dx}.$$

To interpret the Perloff-Salop markup equation, use the notation M_{n-1} (the largest of the $n-1$ noise realizations: $M_{n-1} \equiv \max_{j \in \{1,\dots,n\}, j \neq i} X_j$). Then, $D(p, p; n) = \mathbb{P}(X_i > M_{n-1})$, so

$$D(p, p; n) = \mathbb{E}\left[\overline{F}(M_{n-1})\right]. \tag{19}$$

This formulation emphasizes that the demand for good i is driven by the properties of the right-hand tail of the cumulative distribution function $\bar{F}$, as M_{n-1} is likely to be large.

Sattinger (1984)

Under the utility specification (10), goods from the monopolistically competitive (MC) market are perfect substitutes. The consumer optimizes by buying only one monopolistically competitive good: the good i which maximizes e^{X_i}/p_i. The consumer's utility function is thus Cobb-Douglas in the composite good and the chosen MC good. It is then easy to show that the consumer spends fraction θ of his income on the chosen MC good. Without loss of generality, normalize the consumer's endowment y to equal $1/\theta$, so that the consumer always spends 1 unit of income on the MC good. The demand function of firm i is the probability

that the good i has a higher attraction-price ratio than all other goods, multiplied by the purchased quantity $1/p_i$ of the chosen good i; so

$$D\left(p_1, ..., p_n; i\right) = \frac{1}{p_i}\mathbb{P}\left(\frac{e^{X_i}}{p_i} = \max_{j=1,...,n}\frac{e^{X_j}}{p_j}\right)$$

$$= \frac{1}{p_i}\mathbb{P}\left(X_i - \ln p_i = \max_{j=1,...,n} X_j - \ln p_j\right). \qquad (20)$$

We may rewrite this expression as

$$D\left(p_1, ..., p_n; i\right) = \frac{1}{p_i}\int f(x)\prod_{j\neq i} F\left(x - \ln p_i + \ln p_j\right) dx.$$

Proceeding as in the case of the Perloff-Salop model, we get

$$D\left(p_i, p; n\right) = \frac{1}{p_i}\int_{w_l}^{w_u} f\left(x\right) F^{n-1}\left(x - \ln p_i + \ln p\right) dx,$$

$$D_1\left(p_i, p; n\right) = -\frac{1}{p_i^2}\int_{w_l}^{w_u} f\left(x\right) F^{n-1}\left(x - \ln p_i + \ln p\right) dx$$

$$- \frac{(n-1)}{p_i^2}\int_{w_l}^{w_u} f\left(x\right) f\left(x - \ln p_i + \ln p\right) F^{n-2}\left(x - \ln p_i + \ln p\right) dx$$

In a symmetric equilibrium

$$D(p, p; n) = \int_{w_l}^{w_u} f(x)F^{n-1}(x)\, dx = \frac{1}{pn},$$

$$D_1(p, p; n) = -\frac{1}{p^2}\left(\frac{1}{n} + (n-1)\int_{w_l}^{w_u} f^2(x)F^{n-2}(x)\, dx\right)$$

After some simple calculations, we may show that the Sattinger markup μ_n^{Satt} is

$$p - c = -\frac{D(p, p; n)}{D_1(p, p; n)} = \frac{c}{n(n-1)\int_{w_l}^{w_u} f^2(x)F^{n-2}(x)\, dx}.$$

Hart (1985)

Recall that the consumer's objective is to choose quantities to maximize:

$$\max_{i=1...n}\max_{Q_i\geq 0} U = \frac{\psi+1}{\psi}\left(\sum_{i=1}^{n} e^{X_i} Q_i\right)^{\psi/(\psi+1)} - \sum_{i=1}^{n} p_i Q_i. \qquad (21)$$

3

As in the Sattinger case, it is clear that because goods are perfect substitutes, the consumer will purchase only from one firm, which we denote by i. The first-order condition of the consumer's problem is then

$$0 = \frac{d}{dQ_i}\left[\frac{\psi+1}{\psi}\left(e^{X_i}Q_i\right)^{\psi/(\psi+1)} - p_iQ_i\right] = e^{X_i\psi/(\psi+1)}Q_i^{-1/(\psi+1)} - p_i$$

which gives us the optimal quantity for the chosen good i: $Q_i = e_i^{X_i\psi}/p_i^{1+\psi}$, and the total net utility is:

$$V_i = \frac{\psi+1}{\psi}\left(e^{X_i}Q_i\right)^{\psi/(\psi+1)} - p_iQ_i$$

$$= \left(\frac{\psi+1}{\psi} - 1\right)p_iQ_i = \frac{1}{\psi}p_ie_i^{X_i\psi}/p_i^{1+\psi} = \frac{1}{\psi}\left(\frac{e_i^{X_i}}{p_i}\right)^{\psi}$$

The consumer chooses the good that maximizes his net utility, i.e. $\arg\max_i\left(e^{X_i}/p_i\right)$. We may then calculate the demand function for good i as

$$D(p_1, ..., p_n; i) = \mathbb{E}\left[\frac{e^{\psi X_i}}{p_i^{1+\psi}}I_{\left\{e^{X_i}/p_i=\max_{j=1,...,n}e^{X_j}/p_j\right\}}\right] \tag{22}$$

$$= \mathbb{E}\left[\frac{e^{\psi X_i}}{p_i^{1+\psi}}I_{\{X_i-\ln p_i=\max_{j=1,...,n}X_j-\ln p_j\}}\right] \tag{23}$$

where $I\{\cdot\}$ is the indicator function. Writing out the expectation and differentiating gives

$$D\left(p_i, p; n\right) = \frac{1}{p_i^{1+\psi}}\int_{w_l}^{w_u}e^{\psi x}f\left(x\right)F^{n-1}\left(x - \ln p_i + \ln p\right)dx,$$

$$D_1\left(p_i, p; n\right) = -\frac{1+\psi}{p_i^{2+\psi}}\int_{w_l}^{w_u}e^{\psi x}f\left(x\right)F^{n-1}\left(x - \ln p_i + \ln p\right)dx$$

$$- \frac{n-1}{p_i^{2+\psi}}\int_{w_l}^{w_u}e^{\psi x}f\left(x\right)f\left(x - \ln p_i + \ln p\right)F^{n-2}\left(x - \ln p_i + \ln p\right)dx.$$

In a symmetric equilibrium

$$D(p, p; n) = \frac{1}{p^{1+\psi}}\int_{w_l}^{w_u}e^{\psi x}f\left(x\right)F^{n-1}\left(x\right)dx$$

$$D_1(p, p; n) = -\frac{1}{p^{2+\psi}}\left((1+\psi)\int_{w_l}^{w_u}e^{\psi x}f\left(x\right)F^{n-1}\left(x\right)dx + (n-1)\int_{w_l}^{w_u}e^{\psi x}f^2\left(x\right)F^{n-2}\left(x\right)dx\right).$$

With some simple calculations, we may show that the Hart markup μ_n^{Hart} is

$$p - c = -\frac{D(p, p; n)}{D_1(p, p; n)}$$

$$= c \left(\psi + (n - 1) \frac{\int e^{\psi x} f^2(x) F^{n-2}(x) \, dx}{\int e^{\psi x} f(x) F^{n-1}(x) \, dx} \right)^{-1}.$$

Appendix C   Endogenous Product Differentiation

So far, we have assumed that the variance of the noise term is exogenous. We now relax this assumption and allow firms to choose the degree of product differentiation (in the traditional economic interpretation), or the degree of obfuscation / "confusion" (in a complementary behavioral interpretation). Assume that firms can choose the degree to which their own product is differentiated from the rest of the market; specifically, assume that each firm i can choose σ_i at a cost $c(\sigma_i)$ so that the firm's taste shock is $X_i = \sigma_i X$, where X has CDF F. The game then has the following timing:

1. Firms simultaneously choose (p_i, σ_i)

2. Random taste shocks are realized

3. Consumers make purchase decisions

4. Profits are realized

Firm i's profit function is given by

$$\pi\left((p_i, \sigma_i), (p, \sigma); n\right) = (p_i - c(\sigma_i)) D\left((p_i, \sigma_i), (p, \sigma); n\right)$$

in step 1, where $D\left((p_i, \sigma_i), (p, \sigma); n\right)$ is the demand for good i when the firm chooses (p_i, σ_i) and the remaining $n - 1$ firms choose (p, σ). Each firm i then chooses (p_i, σ_i) to maximize $\pi\left((p_i, \sigma_i), (p, \sigma); n\right)$. The symmetric equilibrium is characterized by

$$(p, \sigma) = \arg\max_{(p', \sigma')} \pi\left((p', \sigma'), (p, \sigma); n\right).$$

Our techniques allow us to analyze the symmetric equilibrium for each of the Perloff-Salop, Sattinger and Hart models.

Proposition C1. *Consider the Perloff-Salop, Sattinger and Hart models where firms simultaneously choose p and σ, under the same assumptions as Theorem 2. Assume that $w_u > 0$.[28] Further, in the Perloff-Salop and Sattinger cases, assume that $xf^2(x)$ is (w_l, w_u)-integrable, and that $c' > 0, c'' > 0, \lim_{t \to \infty} c'(t) = \infty$. In the Hart case, assume that $c' > 0, (\ln c)'' > 0, \lim_{t \to \infty} (\ln c(t))' = \infty$.*

Then the equilibrium outcome with n firms is asymptotically, as $n \to \infty$

$$\mu_n^{PS}(\sigma_n) = \frac{\mu_n^{Satt}(\sigma_n)}{c^{Satt}(\sigma_n)} \sim \frac{\mu_n^{Hart}(\sigma_n)}{c^{Hart}(\sigma_n)} \sim \frac{\sigma_n}{nf\left(F^{-1}\left(1 - \frac{1}{n}\right)\right)\Gamma(\gamma + 2)},$$

$$c^{PS'}(\sigma_n) = \frac{c^{Satt'}(\sigma_n)}{c^{Satt}(\sigma_n)} \sim \frac{c^{Hart'}(\sigma_n)}{c^{Hart}(\sigma_n)} \sim \begin{cases} \overline{F}^{-1}(1/n) : w_u < \infty \\ \frac{\overline{F}^{-1}(1/n)}{\Gamma(\gamma+2)} : w_u = \infty \end{cases}.$$

In other words, at the symmetric equilibrium, the normalized marginal cost of σ – that is $c'(\sigma_n)$ in the Perloff-Salop case and $c'(\sigma_n)/c(\sigma_n)$ in the Sattinger and Hart cases – asymptotically equals $\overline{F}^{-1}\left(\frac{1}{n}\right)$, up to a scaling constant. In particular, the normalized marginal cost of σ goes closer to the upper bound of the distribution as the number of firms increases. Hence, Proposition C1 quantitatively characterizes the monotonic relationship between the number of firms and the degree of endogenous product differentiation (in the traditional economic interpretation), and/or the relationship between the number of firms and the degree of endogenous confusion (in the behavioral interpretation). This effect of competition on the supply of confusion or noise is potentially important in understanding imperfect competition (see e.g., Gabaix and Laibson 2006, Spiegler 2006, Carlin 2009, and Ellison and Ellison 2009). We can use the limit pricing heuristic from Section 2.4 to obtain an intuition for this result. Consider the Perloff-Salop case. Since the firm engages in limit pricing, it can charge a markup of $\sigma M_n - \sigma^* S_n$ where σ is the firm's product differentiation choice and σ^* is the choice of all other firms, which we take as given. The marginal value of an additional unit of noise σ is thus $M_n \simeq \overline{F}^{-1}\left(\frac{1}{n}\right)$.

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[28]This assumption that the largest possible realization of X_i is positive (possibly infinite) makes the firm's problem economically sensible. If, on the contrary, $w_u \leq 0$, then each realization of X_i would be negative with probability 1. In that case, increasing σ would *reduce* the attractiveness of the firm's product to the consumer. To eliminate this possibility, we assume $w_u > 0$.

Appendix D   Proofs Omitted from the Paper

Proof of Lemma A1.

1. Follows by inversion of Resnick (1987, Prop. 0.5).

2. This fact follows from the observation that for $\frac{g(s(x))}{g(h(x))} = \frac{g\left(\frac{s(x)}{h(x)}h(x)\right)}{g(h(x))} \sim \left(\frac{s(x)}{h(x)}\right)^{\rho} \to_{x\to 0} 1$ where we can take the limit as $x \to 0$ because of Lemma A1.1. Going into more detail, choose $\delta(\cdot)$ such that $\lim_{t\to 0} \delta(t) = 0$ and $|s(t')/h(t') - 1| < \delta(t)$ for $t' < t$. Such $\delta(\cdot)$ exists by our assumptions on s and h. Choose $\varepsilon(\cdot, \cdot)$ such that $\lim_{t\to 0} \varepsilon(t, \delta) = \lim_{\delta\to 0} \varepsilon(t, \delta) = 0$ and $|g(xt')/g(t') - x^{\rho}| < \varepsilon(t, \delta)$ for $x \in (1 - \delta, 1 + \delta)$ and $t' < t$. Lemma A1.1 ensures that such $\varepsilon(\cdot, \cdot)$ exists. Then

$$|g(s(t'))/g(h(t')) - 1| = \left|g\left(\frac{s(t')}{h(t')}h(t')\right)/g(h(t')) - 1\right| < \varepsilon(h(t'), \delta(t)) + \rho O(\delta(t))$$

for $t' < t$. Since the RHS goes to zero as $t \to 0$, the result follows.

3. Since $\lim_{t\to 0}\frac{g(xt)}{g(t)} = x^a$ and $\lim_{t\to 0}\frac{h(xt)}{h(t)} = x^b$, we have $\lim_{t\to 0}\frac{g(xt)h(xt)}{g(t)h(t)} = x^{a+b}$.

4. Follows by inversion of Resnick (1987, Prop. 0.8, iv).

5. Follows by inversion of Resnick (1987, Prop. 0.8, v).

6. Both parts follow by inversion of Resnick (1987, Th. 0.6, a).

7. Follows from Resnick (1987, Prop. 0.7) and by inversion.

8. Follows from Resnick (1987, Prop. 0.8, ii) and by inversion.

$\square$

To prove Proposition 2, we introduce a lemma that links differences between the two top order statistics to the behavior of the top tail statistics, and hence allows us to apply our general results.

Lemma D2. *Call M_n and S_n, respectively, the largest and second largest realizations of n i.i.d. random variables with CDF F and density $f = F'$, and G a function such that $\int G(x) f(x)\, dx < \infty$, $\lim_{x\to F^{-1}(0)} G(x) F(x) = \lim_{x\to F^{-1}(1)} G(x) \overline{F}(x) = 0$. Then:*

$$\mathbb{E}\left[G(M_n) - G(S_n)\right] = \mathbb{E}\left[\frac{G'(M_n)\overline{F}(M_n)}{f(M_n)}\right] \tag{24}$$

Proof. Recall that the density of M_n is $nf(x)F^{n-1}(x)$, and the density of S_n is

$$n(n-1)f(x)\overline{F}(x)F^{n-2}(x).$$

So

$$\mathbb{E}[G(S_n)] = \int n(n-1)G(x)f(x)\overline{F}(x)F^{n-2}(x)\,dx$$

$$= n\left[G(x)\overline{F}(x)F^{n-1}(x)\right]_{F^{-1}(0)}^{F^{-1}(1)} - \int n\left(G(x)\overline{F}(x)\right)' F^{n-1}(x)\,dx$$

$$= 0 + \int nG(x)f(x)F^{n-1}(x)\,dx - \int n\frac{G'(x)\overline{F}(x)}{f(x)}f(x)F^{n-1}(x)\,dx$$

$$= \mathbb{E}[G(M_n)] - \mathbb{E}\left[\frac{G'(M_n)\overline{F}(M_n)}{f(M_n)}\right] \qquad \square$$

Proof of Proposition 2. Proposition 2 is simply an application of Lemma D2 to the special case $G(x) = x$. As $f(\overline{F}^{-1}(t)) \in RV_{1+\gamma}^0$, $t/f(\overline{F}^{-1}(t)) \in RV_{-\gamma}^0$, and we may apply Theorem 3 to obtain the desired result. $\qquad \square$

Proof of Theorem 2. As with Theorem 1, the Sattinger case follows immediately from Proposition 3. We omit those calculations and focus on the Hart case. Applying Proposition 3 to (13), we immediately infer that

$$\frac{\mu_n^{Hart}}{c} \sim \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right)\frac{\Gamma(\gamma+2-\psi a)}{\Gamma(1-\psi a)}}$$

under the conditions of the theorem. We will use the fact that $anf\left(\overline{F}^{-1}(1/n)\right) \sim 1$, which holds because

$$\lim_{n\to\infty} \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)} = \lim_{x\to w_u} \frac{\overline{F}(x)}{f(x)} = a.$$

Consider first the case where $a = 0$. Then $nf\left(\overline{F}^{-1}(1/n)\right) \to \infty$, and the expression simplifies to

$$\frac{\mu_n^{Hart}}{c} \sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\left[\frac{\psi}{nf\left(\overline{F}^{-1}(1/n)\right)} + \frac{\Gamma(\gamma+2-\psi a)}{\Gamma(1-\psi a)}\right]} \sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\Gamma(\gamma+2)}.$$

8

Next, consider the case $0 < a < \infty$, which implies $\gamma = 0$. We have

$$\frac{\mu_n^{Hart}}{c} \sim \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right) \frac{\Gamma(2-\psi a)}{\Gamma(1-\psi a)}} = \frac{1}{\psi + nf\left(\overline{F}^{-1}(1/n)\right)(1-\psi a)}$$

$$= \frac{1}{\psi\left(1 - anf\left(\overline{F}^{-1}(1/n)\right)\right) + nf\left(\overline{F}^{-1}(1/n)\right)}$$

$$\sim \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)} = \frac{1}{nf\left(\overline{F}^{-1}(1/n)\right)\Gamma(2+\gamma)}$$

when $\gamma = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition C1. First, some notation: $\pi\left((p,\sigma),(p^*,\sigma^*);n\right)$ denotes the profit function of a firm that chooses (p,σ) when the remaining $n-1$ firms choose (p^*,σ^*). Also, $\pi(p,\sigma;n)$ denotes the profit function of a firm when all n firms choose (p,σ).

Perloff-Salop Case Call σ^* and p^* the equilibrium choices of the other firms:

$$\pi\left((p,\sigma),(p^*,\sigma^*);n\right) = (p-c(\sigma))\,\mathbb{P}\left(\sigma X_1 - p \geq \max_{j\neq i}\sigma^* X_j - p^*\right)$$

$$= (p-c(\sigma))\,\mathbb{P}\left(\frac{\sigma}{\sigma^*}X_i + \frac{p^*-p}{\sigma^*} \geq \max_{j\neq i}X_j\right)$$

$$= (p-c(\sigma))\int f(x)\,F^{n-1}\left(\frac{\sigma}{\sigma^*}x + \frac{p^*-p}{\sigma^*}\right)dx.$$

The first-order conditions for profit maximization are as follows. Differentiating with respect to p yields

$$p - c(\sigma) = \frac{\int f(x)\,F^{n-1}(x)\,dx}{\frac{1}{\sigma}(n-1)\int f^2(x)\,F^{n-2}(x)\,dx}$$

and differentiating with respect to σ gives

$$c'(\sigma)\int f(x)\,F^{n-1}(x)\,dx = (n-1)(p-c(\sigma))\int xf^2(x)\,F^{n-2}(x)\,dx\frac{1}{\sigma}.$$

Some manipulation reveals

$$c'(\sigma) = \frac{\int xf^2(x)\,F^{n-2}(x)\,dx}{\int f^2(x)\,F^{n-2}(x)\,dx}.$$

9

Now we consider two cases: $w_u < \infty$ and $w_u = \infty$. If $w_u < \infty$, then

$$\frac{\int x f^2 (x) F^{n-2} (x) \, dx}{\int f^2 (x) F^{n-2} (x) \, dx} = \frac{n^{-1} w_u f \left(\overline{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)}{n^{-1} f \left(\overline{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)} + o (1) = w_u + o (1) .$$

If $w_u = \infty$ then

$$\frac{\int x f^2 (x) F^{n-2} (x) \, dx}{\int f^2 (x) F^{n-2} (x) \, dx} \sim \frac{n^{-1} \overline{F}^{-1} (1/n) f \left(\overline{F}^{-1} (1/n) \right) \Gamma (2)}{n^{-1} f \left(\overline{F}^{-1} (1/n) \right) \Gamma (\gamma + 2)} \sim \frac{\overline{F}^{-1} (1/n)}{\Gamma (\gamma + 2)} .$$

Sattinger Case   We have

$$\pi \left((p, \sigma), (p^*, \sigma^*) ; n \right) = \frac{p - c (\sigma)}{p} \mathbb{P} \left(\frac{e^{\sigma X_i}}{p} \geq \max_{j \neq i} \frac{e^{\sigma^* X_j}}{p^*} \right)$$

$$= \frac{p - c (\sigma)}{p} \int f (x) F^{n-1} \left(\frac{\sigma}{\sigma^*} x + \frac{\log p^* - \log p}{\sigma^*} \right) dx$$

so the first-order conditions for profit maximization become

$$0 = \pi_2 (p, \sigma; n) = -\frac{c' (\sigma)}{p} \int f (x) F^{n-1} (x) \, dx + \frac{p - c (\sigma)}{\sigma p} (n - 1) \int x f^2 (x) F^{n-2} (x) \, dx$$

and

$$0 = \pi_1 (p, \sigma; n) = \frac{c (\sigma)}{p^2} \int f (x) F^{n-1} (x) \, dx - \frac{p - c (\sigma)}{\sigma p^2} (n - 1) \int f^2 (x) F^{n-2} (x) \, dx$$

Rearranging, we get

$$\frac{p - c (\sigma)}{c (\sigma)} = \frac{\sigma}{n (n - 1) \int f^2 (x) F^{n-2} (x) \, dx}$$

and

$$c' (\sigma) = \frac{\frac{p - c(\sigma)}{\sigma p} (n - 1) \int x f^2 (x) F^{n-2} (x) \, dx}{\int f (x) F^{n-1} (x) \, dx} ,$$

so

$$\frac{c'(\sigma)}{c(\sigma)} = \frac{\int x f^2(x) F^{n-2}(x)\, dx}{\int f^2(x) F^{n-2}(x)\, dx}$$

$$= \begin{cases} \overline{F}^{-1}(1/n) + o\left(\overline{F}^{-1}(1/n)\right) = w_u + o(1) : w_u < \infty \\[2mm] \frac{\overline{F}^{-1}(1/n) + o\left(\overline{F}^{-1}(1/n)\right)}{\Gamma(\gamma+2)} : w_u = \infty \end{cases}$$

as calculated in the Perloff-Salop case.

Hart Case   We have

$$\pi\left((p,\sigma),(p^*,\sigma^*);n\right) = (p - c(\sigma))\, \mathbb{E}\left[\frac{e^{\psi\sigma X_i}}{p^{1+\psi}} I_{\left\{\frac{e^{\sigma X_i}}{p} \geq \max_{j \neq i} \frac{e^{\sigma^* X_j}}{p^*}\right\}}\right]$$

$$= (p - c(\sigma))\, \mathbb{E}\left[\frac{e^{\psi\sigma X_i}}{p^{1+\psi}} I_{\left\{\frac{\sigma}{\sigma^*} X_i + \frac{\log p^* - \log p}{\sigma^*} = \max_{j \neq i} X_j\right\}}\right]$$

$$= (p - c(\sigma)) \int \frac{e^{\psi\sigma x}}{p^{1+\psi}} f(x) F^{n-1}\left(\frac{\sigma}{\sigma^*} x + \frac{\log p^* - \log p}{\sigma^*}\right) dx$$

so the first-order conditions for profit maximization become

$$0 = \pi_2(p,\sigma;n) = -c'(\sigma) \int \frac{e^{\psi\sigma x}}{p^{1+\psi}} f(x) F^{n-1}(x)\, dx + (p - c(\sigma)) \left\{ \begin{array}{l} \int \psi x \frac{e^{\psi\sigma x}}{p^{1+\psi}} f(x) F^{n-1}(x)\, dx \\[2mm] + \frac{n-1}{\sigma} \int x \frac{e^{\psi\sigma x}}{p^{1+\psi}} f^2(x) F^{n-2}(x)\, dx \end{array} \right\}$$

and

$$0 = \pi_1(p,\sigma;n) = \int \frac{e^{\psi\sigma x}}{p^{1+\psi}} f(x) F^{n-1}(x)\, dx - (p - c(\sigma)) \left\{ \begin{array}{l} (1+\psi) \int \frac{e^{\psi\sigma x}}{p^{2+\psi}} f(x) F^{n-1}(x)\, dx \\[2mm] + \frac{n-1}{\sigma} \int \frac{e^{\psi\sigma x}}{p^{2+\psi}} f^2(x) F^{n-2}(x)\, dx \end{array} \right\}$$

so

$$\frac{p - c(\sigma)}{c(\sigma)} = \frac{\int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx}{\psi \int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}$$

and

$$\frac{c'(\sigma)}{c(\sigma)} = \frac{p - c(\sigma)}{c(\sigma)} \frac{\int \psi x e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int x e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}{\int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx}$$

$$= \frac{\psi \int x e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int x e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}{\psi \int e^{\psi\sigma x} f(x) F^{n-1}(x)\, dx + \frac{n-1}{\sigma} \int e^{\psi\sigma x} f^2(x) F^{n-2}(x)\, dx}.$$

11

Now we consider two cases: $w_u < \infty$ and $w_u = \infty$. If $w_u < \infty$, then (noting that $a = 0$ in this case)

$$
\begin{aligned}
\frac{c'(\sigma)}{c(\sigma)} &= \frac{\psi \int xe^{\psi\sigma x}f(x)F^{n-1}(x)\,dx + \frac{n-1}{\sigma}\int xe^{\psi\sigma x}f^2(x)F^{n-2}(x)\,dx}{\psi \int e^{\psi\sigma x}f(x)F^{n-1}(x)\,dx + \frac{n-1}{\sigma}\int e^{\psi\sigma x}f^2(x)F^{n-2}(x)\,dx} \\
&= \frac{\psi n^{-1}w_u e^{\sigma\psi w_u}\Gamma(1) + \frac{1}{\sigma}w_u e^{\sigma\psi w_u}f\left(\overline{F}^{-1}(1/n)\right)\Gamma(\gamma+k)}{\psi n^{-1}e^{\sigma\psi w_u}\Gamma(1) + \frac{1}{\sigma}e^{\sigma\psi w_u}f\left(\overline{F}^{-1}(1/n)\right)\Gamma(\gamma+k)} + o(1) \\
&= w_u + o(1).
\end{aligned}
$$

If $w_u = \infty$, then noting that $\gamma = 0$ and denoting $U_n = \overline{F}^{-1}(1/n)$,

$$
\begin{aligned}
\frac{c'(\sigma)}{c(\sigma)} &= \frac{\psi \int xe^{\psi\sigma x}f(x)F^{n-1}(x)\,dx + \frac{n-1}{\sigma}\int xe^{\psi\sigma x}f^2(x)F^{n-2}(x)\,dx}{\psi \int e^{\psi\sigma x}f(x)F^{n-1}(x)\,dx + \frac{n-1}{\sigma}\int e^{\psi\sigma x}f^2(x)F^{n-2}(x)\,dx} \\
&\sim \frac{\psi n^{-1}U_n e^{\sigma\psi U_n}\Gamma(1-\psi a) + \frac{1}{\sigma}\psi U_n f(U_n)e^{\sigma\psi U_n}\Gamma(2-\psi a)}{\psi n^{-1}e^{\sigma\psi U_n}\Gamma(1-\psi a) + \frac{1}{\sigma}\psi f(U_n)e^{\sigma\psi U_n}\Gamma(2+\gamma-\psi a)} \\
&= U_n. \qquad\qquad \square
\end{aligned}
$$

Appendix E   Second-Order Conditions for Profit Maximization

Recall that the profit function $\pi(p_i, p)$ for firm i when it sets price p_i and all other firms set price p is

$$
\pi(p_i, p) = (p_i - c)D(p_i, p) - K. \tag{25}
$$

So far, we have analyzed the first-order condition for profit maximization, $\pi_1(p, p; n) = 0$, which is necessary but not sufficient to ensure equilibrium. Anderson, De Palma, and Thisse (1992) show (Prop. 6.5, p.171 and Prop. 6.9, p.184) that symmetric price equilibria exist in the Perloff-Salop, Sattinger and Hart models when f is log-concave. Thus in these cases (25) defines the unique symmetric price equilibrium. However, their results do not cover distributions where f is not log-concave. We are unable to derive global conditions for existence of equilibrium in these cases. Instead, we verify in this appendix that the markups we study satisfy the second-order conditions for profit-maximization. The following three propositions show that the symmetric equilibrium markup expression (3) which we use in our calculations

also satisfies the second-order condition for profit maximization, $\pi_{11}(p, p; n) < 0$. It is useful to note that, via simple calculations, the second order condition is

$$\pi_{11}(p, p; n) = 2D_1(p, p; n) - \frac{D(p, p; n)}{D_1(p, p; n)}D_{11}(p, p; n) < 0. \tag{26}$$

Proposition E2. *Assume that F satisfies the conditions for Theorem 1, that $f^3(x)$ is (w_l, w_u)-integrable, and that*

$$-4\Gamma(\gamma + 2)^2 + \Gamma(2\gamma + 3) < 0,$$

which holds for $-1.45 < \gamma < 0.64$. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Perloff-Salop model.

Note that Proposition E2 covers all distributions with thin $(-1 \leq \gamma \leq 0)$ and medium fat tails $(\gamma = 0)$, and all the heavy tailed distributions with a finite variance, i.e. $\gamma \in (0, 1/2]$.

Proposition E3. *Assume that F satisfies the conditions for Theorem 1, that $f^3(x)$ is (w_l, w_u)-integrable, and that either $\gamma > 0$ or*

$$-4\Gamma(\gamma + 2)^2 + \Gamma(2\gamma + 3) < 0,$$

which holds for $-1.45 < \gamma \leq 0$. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Sattinger model.

Proposition E4. *Assume that the conditions for Theorem 1 are satisfied, and that $e^{\psi x}f^3(x)$ is (w_l, w_u)-integrable. Then the second-order condition for profit maximization is satisfied in the symmetric equilibrium of the Hart model.*

Proof of Proposition E2. We denote $U_n = \overline{F}^{-1}(1/n)$ in several of the proofs below. Note, from Section B in this Appendix, that

$$D(p_i, p) = \int f(x) F^{n-1}(x + p - p_i) \, dx \text{ and}$$

$$D_1(p_i, p) = -(n-1)\int f(x) f(x + p - p_i) F^{n-2}(x + p - p_i) \, dx,$$

from which we may calculate

$$D_{11}(p, p) = \frac{(n-1)(n-2)}{2}\int f^3(x) F^{n-3}(x) \, dx + \frac{n-1}{2}f^2(x) F^{n-2}(x)\Big|_{-\infty}^{\infty}$$

13

where the last term on the RHS vanishes. So, applying Proposition 3,

$$\pi_{11}\left(p,p;n\right) = 2D_1\left(p,p;n\right) - \frac{D\left(p,p;n\right)}{D_1\left(p,p;n\right)}D_{11}\left(p,p;n\right)$$

$$= -2\left(n-1\right)\int f^2\left(x\right)F^{n-2}\left(x\right)\,dx + \frac{\frac{(n-1)(n-2)}{2}\int f^3\left(x\right)F^{n-3}\left(x\right)\,dx}{n\left(n-1\right)\int f^2\left(x\right)F^{n-2}\left(x\right)\,dx}$$

$$= -2\left(n-1\right)\int f^2\left(x\right)F^{n-2}\left(x\right)\,dx + \frac{\left(n-2\right)\int f^3\left(x\right)F^{n-3}\left(x\right)\,dx}{2n\int f^2\left(x\right)F^{n-2}\left(x\right)\,dx}$$

$$\sim -2f\left(U_n\right)\Gamma\left(\gamma+2\right) + \frac{f\left(U_n\right)\Gamma\left(2\gamma+3\right)}{2\Gamma\left(\gamma+2\right)}$$

$$= \frac{f\left(U_n\right)}{2\Gamma\left(\gamma+2\right)}\left(-4\Gamma\left(\gamma+2\right)^2 + \Gamma\left(2\gamma+3\right)\right).$$

We can easily verify numerically that $-4\Gamma\left(\gamma+2\right)^2 + \Gamma\left(2\gamma+3\right) < 0$ for $-1.45 < \gamma \le 0$; it follows that

$$\pi_{11}\left(p,p;n\right) < 0 \text{ for } \gamma \in \left[-1.45, 0.64\right]. \qquad \square$$

Proof of Proposition E3. Without loss of generality, let $\theta y = 1$. Then, from Section B in this Appendix,

$$D\left(p_i,p\right) = \frac{1}{p_i}\int f\left(x\right)F^{n-1}\left(x+\ln p - \ln p_i\right)\,dx \text{ and}$$

$$D_1\left(p_i,p\right) = -\frac{1}{p_i^2}\int f\left(x\right)F^{n-1}\left(x+\ln p - \ln p_i\right)\,dx$$
$$- \frac{n-1}{p_i^2}\int f\left(x\right)f\left(x+\ln p - \ln p_i\right)F^{n-2}\left(x+\ln p - \ln p_i\right)\,dx,$$

from which we may calculate

$$D_{11}\left(p,p\right) = \frac{2}{p^3}\int f\left(x\right)F^{n-1}\left(x\right)\,dx + 3\frac{n-1}{p^3}\int f^2\left(x\right)F^{n-2}\left(x\right)\,dx$$
$$+ \frac{\left(n-1\right)\left(n-2\right)}{2p^3}\int f^3\left(x\right)F^{n-3}\left(x\right)\,dx + \frac{n-1}{2p^3}\left[f^2\left(x\right)F^{n-2}\left(x\right)\right]_{-\infty}^{\infty}$$

where the last term on the RHS vanishes. We may then substitute our expressions for $D\left(p,p;n\right), D_1\left(p,p;n\right), D_{11}\left(p,p;n\right)$ into (26) and apply Proposition 3. The asymptotic ex-

14

pression simplifies to

$$\pi_{11}(p,p;n) = 2D_1(p,p;n) - \frac{D(p,p;n)}{D_1(p,p;n)}D_{11}(p,p;n)$$

$$= -\frac{2}{p^2}\left(\int f(x)F^{n-1}(x)\,dx + (n-1)\int f^2(x)F^{n-2}(x)\,dx\right)$$

$$+ \frac{\left(\begin{array}{c}2\int f(x)F^{n-1}(x)\,dx + 3(n-1)\int f^2(x)F^{n-2}(x)\,dx \\ +\frac{(n-1)(n-2)}{2}\int f^3(x)F^{n-3}(x)\,dx\end{array}\right)}{p^2 n\left(\int f(x)F^{n-1}(x)\,dx + (n-1)\int f^2(x)F^{n-2}(x)\,dx\right)}$$

$$= \frac{p^{-2}}{n}\left(\begin{array}{c}-2(1+nf(U_n)\Gamma(\gamma+2)) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))}\end{array}\right)$$

In the case $nf(U_n) = o(1)$, which implies $\gamma \geq 0$ and $f(w_u) = 0$, we get

$$\pi_{11}(p,p;n) = \frac{p^{-2}}{n}\left(\begin{array}{c}-2(1+nf(U_n)\Gamma(\gamma+2)) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+\frac{1}{2}(nf(w_u))^2+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))}\end{array}\right)$$

$$= \frac{p^{-2}}{n}\left(\begin{array}{c}-2(1+nf(U_n)\Gamma(\gamma+2)) + \frac{2+3nf(U_n)\Gamma(\gamma+2)}{1+nf(U_n)\Gamma(\gamma+2)} \\ +o(nf(U_n))\end{array}\right)$$

$$= \frac{p^{-2}}{n}\left(\begin{array}{c}-\frac{nf(U_n)\Gamma(\gamma+2)}{1+nf(U_n)\Gamma(\gamma+2)} \\ +o(nf(U_n))\end{array}\right)$$

$$< 0.$$

In the case $\lim_{n\to\infty} nf(U_n) \in (0,\infty)$, which implies $\gamma = 0$, we get

$$\pi_{11}(p,p;n) = \frac{p^{-2}}{n}\left(\begin{array}{c}-2(1+nf(U_n)\Gamma(\gamma+2)) + o(nf(U_n)) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+\frac{1}{2}(nf(w_u))^2+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))}\end{array}\right)$$

$$= \frac{p^{-2}}{n}\left(-2(1+nf(U_n)) + \frac{2+3nf(U_n)+(nf(U_n))^2}{1+nf(U_n)} + o(nf(U_n))\right)$$

$$= \frac{p^{-2}}{n}\left(-nf(U_n) + o(nf(U_n))\right)$$

$$< 0.$$

In the case $\lim_{n\to\infty} nf(U_n) = \infty$, which implies $\gamma \leq 0$, we get

$$
\begin{aligned}
\pi_{11}(p,p;n) &= \frac{p^{-2}}{n}\left(\begin{array}{c} -2\left(1 + nf(U_n)\,\Gamma(\gamma+2)\right) + o\left(nf(U_n)\right) \\ +\frac{2+3nf(U_n)(\Gamma(\gamma+2))+\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))+o(nf(U_n))^2}{(1+nf(U_n)\Gamma(\gamma+2))+o(nf(U_n))} \end{array}\right) \\
&= \frac{p^{-2}}{n}\left(\begin{array}{c} -2\left(nf(U_n)\,\Gamma(\gamma+2)\right) + o\left(nf(U_n)\right) \\ +\frac{\frac{1}{2}(nf(U_n))^2\Gamma(2\gamma+3)+o(nf(U_n))^2}{nf(U_n)\Gamma(\gamma+2)+o(nf(U_n))} \end{array}\right) \\
&= p^{-2}f(U_n)\left(-2\Gamma(\gamma+2) + \frac{1}{2}\frac{\Gamma(2\gamma+3)}{\Gamma(\gamma+2)}\right);
\end{aligned}
$$

since we can easily verify numerically that $-2\Gamma(\gamma+2) + \frac{1}{2}\frac{\Gamma(2\gamma+3)+1}{\Gamma(\gamma+2)} < 0$ for $-1.45 < \gamma \leq 0$,
it follows that

$$
\pi_{11}(p,p;n) < 0 \text{ for } \gamma \in [-1.45, 0]. \qquad \square
$$

Proof of Proposition E4. Note that in the Hart case, we are restricted to $\gamma \in [-1,0]$. We have, from Section B in this Appendix,

$$
D(p_i,p) = \frac{1}{p_i^{1+\psi}}\int e^{\psi x} f(x) F^{n-1}(x+\ln p - \ln p_i)\, dx \text{ and}
$$

$$
D_1(p_i,p) = -\frac{1}{p_i^{2+\psi}}\left\{\begin{array}{c} (1+\psi)\int e^{\psi x} f(x) F^{n-1}(x+\ln p - \ln p_i)\, dx \\ +(n-1)\int e^{\psi x} f(x) f(x+\ln p - \ln p_i) F^{n-2}(x+\ln p - \ln p_i)\, dx \end{array}\right\},
$$

from which we may calculate

$$
D_{11}(p,p) = \frac{1}{p^{3+\psi}}\left\{\begin{array}{c} (1+\psi)(2+\psi)\int e^{\psi x} f(x) F^{n-1}(x)\, dx \\ +3\left(1+\frac{\psi}{2}\right)(n-1)\int e^{\psi x} f^2(x) F^{n-2}(x)\, dx \\ +\frac{1}{2}(n-1)(n-2)\int e^{\psi x} f^3(x) F^{n-3}(x)\, dx \end{array}\right\}.
$$

We may then substitute our expressions for $D(p,p;n)$, $D_1(p,p;n)$, $D_{11}(p,p;n)$ into (26) and apply Proposition 3. This gives us

$$
2D_1(p,p;n) - \frac{D(p,p;n)}{D_1(p,p;n)}D_{11}(p,p;n) = \frac{e^{\psi U_n}}{p_i^{2+\psi}}(A+B),
$$

16

where

$$A \sim -2\left(1+\psi\right)\Gamma\left(1-a\psi\right) - 2nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right), \text{ and}$$

$$B \sim \Gamma\left(1-a\psi\right)\frac{\left\{\begin{array}{c}\left(1+\psi\right)\left(2+\psi\right)\Gamma\left(1-a\psi\right)\\+3\left(1+\frac{\psi}{2}\right)nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right)\\+\frac{1}{2}\left(nf\left(U_n\right)\right)^2\Gamma\left(2\gamma+3-a\psi\right)\end{array}\right\}}{\left(1+\psi\right)\Gamma\left(1-a\psi\right)+nf\left(U_n\right)\Gamma\left(\gamma+2-a\psi\right)}$$

After some tedious but straightforward calculations: if $a = 0$, then $nf\left(U_n\right) \to_{n\to\infty} \infty$, and the asymptotic expression simplifies to

$$\pi_{11}\left(p,p;n\right)$$
$$\sim \frac{e^{\psi U_n}}{p_i^{2+\psi}}nf\left(U_n\right)\left(-2\Gamma\left(\gamma+2\right) + \frac{\Gamma\left(2\gamma+3\right)}{2\Gamma\left(\gamma+2\right)}\right)$$
$$< 0 \text{ for } \gamma \in [-1,0]$$

Since we can verify that $-2\Gamma\left(\gamma+2\right) + \frac{\Gamma(2\gamma+3)}{2\Gamma(\gamma+2)} < 0$ for $\gamma \in [-1,0]$, our claim holds in the case $a = 0$. If $0 < a < \infty$, then $\gamma = 0$, $nU_n \to 1/a$ and the asymptotic expression simplifies to

$$\pi_{11}\left(p,p;n\right) \sim \frac{e^{\psi U_n}}{p_i^{2+\psi}}\Gamma\left(1-a\psi\right)\left(-2\left(1+1/a\right) + \frac{\left\{\begin{array}{c}\left(1+\psi\right)\left(2+\psi\right)+3\left(1+\psi/2\right)\left(1/a-\psi\right)\\+\frac{1}{2}\left(2/a-\psi\right)\left(1/a-\psi\right)\end{array}\right\}}{1+1/a}\right)$$

$$= -\frac{e^{\psi U_n}}{p_i^{2+\psi}}\frac{\Gamma\left(1-a\psi\right)}{a} < 0. \qquad \square$$

17